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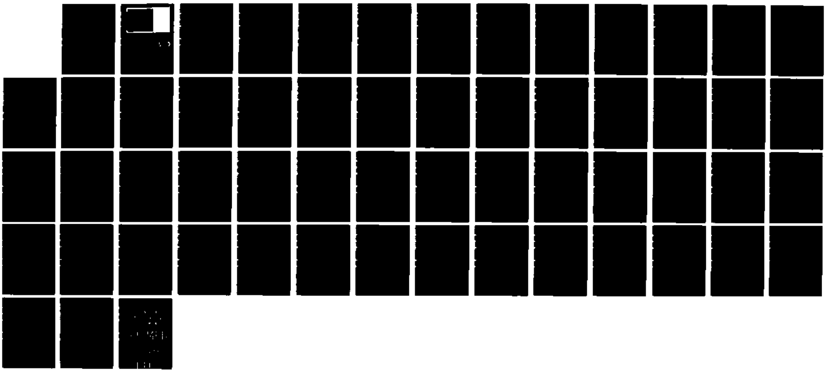

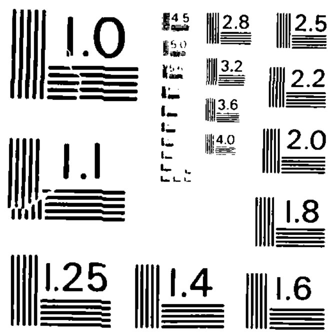

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A STATISTICAL ANALYSIS OF THE EFFECT OF FREESTREAM TURBULENCE OF THE BLASIUS BOUNDARY LAYER

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October 1985

(Received October 8, 1985)

Sponsored by

U.S. Army Research Office
P.O. Box 12211
Research Triangle Park
North Carolina 27709

National Science Foundation
Washington, D.C. 20550

86 2 5 033

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MATHEMATICS RESEARCH CENTER

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ABSTRACT

This paper considers the flow past a flat plate. When the flow is uniform the Blasius boundary layer forms on the plate. Herein the usual uniform free stream is perturbed by a small amplitude field of random three-dimensional turbulence. Using a numerical technique the statistical response in the boundary layer due to the turbulence in the free stream is computed in terms of auto- and cross-correlation functions. The field variables are expanded in finite Chebyshev series and the space-time correlations of these functions are found by forming correlation matrices with the vectors of random coefficients of the Chebyshev expansions.

The main result of the evaluations is the fact that three dimensional random infinitesimal free stream turbulence produces a fully three dimensional response in the boundary layer. The excited field includes the presence of streamwise vorticity as well which was previously thought to be due to secondary instability.

AMS (MOS) Subject Classifications: 15A18, 15A15, 70L05, 76D10

Key Words: boundary layers, turbulence, statistics, Chebyshev polynomials

Work Unit Number 2 (Physical Mathematics)

Sponsored by the United States Army under Contract No. DAAG29-80-C-0041. This material is based upon work supported by the National Science Foundation under Grant No. DMS-8210950, Mod. 1.

A STATISTICAL ANALYSIS OF THE EFFECT OF FREESTREAM TURBULENCE OF THE BLASIUS BOUNDARY LAYER

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1. Introduction

The subject of this paper is an analysis of the response of a boundary layer (the Blasius boundary layer) where the free stream contains random three-dimensional turbulence of small amplitude. The statistics of the free stream turbulence are considered known, and from these and the governing equations a statistical solution is found in the boundary layer.

Previus theoretical work on this problem up until 1975 has been summarized by Rogler and Reshotko (1975) and this may be found in their paper. In their work Rogler and Reshotko (1974, 1975) studied the problem in question when the free stream disturbances are deterministic and two-dimensional. The free stream is composed of a uniform flow plus a set of inviscid point vortices which convect with the mean flow. The analysis is limited by the two-dimensionality but their results show a strong interaction between the boundary layer and the free stream with excited amplitudes of the boundary layer vorticity reaching twice the free stream value. Their result also shows that there is a correlation between the length scale of the point vortices and the distance from the leading edge.

In this paper the work of Rogler and Reshotko is extended to include a three dimensional analysis and the free stream disturbances are taken to be a field of random three dimensional turbulence. This free stream turbulence is three-dimensional in space, is time dependent, and is composed of a fluctuating velocity and pressure field of small amplitude. The properties and characteristics of the random free stream turbulence are identified by specifying a multi-dimensional autocorrelation function. This autocorrelation function includes length and time scales as well as spectral information.

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The response is composed of the Blasius solution plus a perturbation in the boundary layer due to the presence of external free stream turbulence. Since the forcing function is random, the excited response will also be random. Therefore, the solution is determined in terms of auto- and cross-correlations of the excited velocity, pressure, and vorticity fields.

This analysis can be considered a natural extension of the theory of hydrodynamic stability. In hydrodynamic stability theory, qualitative solutions are obtained for a wide range of parameters. By treating the boundary layer as a multi-degree-of-freedom vibrating system, the proper linear combination of these natural modes of the boundary layer can be summed. This results in a statistical solution in the boundary layer which is due to a particular random forcing function in the free stream.

Before proceeding to the details of the analysis of the three-dimensional boundary layer, the background for the analysis can be illustrated by considering the simple analogous problem of a linear SDOF system under stochastic loading. Such a system has the following governing equation and boundary conditons..

$$L\varphi = q(t) \tag{1.1}$$

$$\varphi(0) = \varphi'(0) = 0 \tag{1.2}$$

where

$$L\varphi \equiv \frac{d^2\varphi}{dt^2} + 2\beta\omega_0 \frac{d\varphi}{dt} + \omega_0{}^2\varphi \tag{1.3}$$

β is a damping parameter, and ω_0 is the undamped natural frequency.

In the first analysis, the homogeneous operator is studied. The eigenproperties can be used to form a transfer function which is used utimately to construct the complete solution. By taking the Fourier transform of (1.3), the result is

$$[-\omega^2 - 2i\beta\omega_0\omega + \omega_0{}^2]\hat{\varphi}(\omega) = 0 \tag{1.4}$$

and

$$\omega = \omega_0[\pm\sqrt{1 - \beta^2} - i\beta] \tag{1.5}$$

The statistical properties of the random forcing function must be defined based on some physical observation. Taking $q(t)$ to be statistically stationary an autocorrelation can be defined,

$$R_{qq}(\tau) = \langle q^*(t), q(t+\tau)\rangle \tag{1.6}$$

where the brackets denote an ensemble average and * denotes complex conjugate. Alternatively a spectral density function $S_{qq}(\omega)$ may be defined, which is related to $R_{qq}(\tau)$ by the Fourier transform. If

$$q(t) = \int_{-\infty}^{\infty} \hat{q}(\omega)e^{-i\omega t}d\omega \tag{1.7}$$

then

$$\langle \hat{q}^*(\omega_1), \hat{q}(\omega_2)\rangle = S_{qq}(\omega_1)\delta(\omega_2 - \omega_1) \tag{1.8}$$

where the delta function arises because $q(t)$ is stationary.

The next step is to obtain a solution for $\varphi(t)$ in time. Taking a Fourier transform of $L\varphi = q$ and defining

$$F(\omega) = \left[-\omega^2 - 2i\beta\omega_0\omega + \omega_0^{\,2}\right]^{-1} \tag{1.9}$$

then the solution is

$$\varphi(t) = \int_{-\infty}^{\infty} F(\omega)\hat{q}(\omega)e^{-i\omega t}d\omega \tag{1.10}$$

However only the statistical properties of $\hat{q}(\omega)$ are known. Therefore an autocorrelation function for $\varphi(t)$ must be obtained,

$$R_{\varphi\varphi}(\tau) = \langle \varphi^*(t), \varphi(t+\tau)\rangle \tag{1.11}$$

Substituting (1.10) into (1.11) and using (1.8) the result is

$$R_{\varphi\varphi}(\tau) = \int_{-\infty}^{\infty} |F(\omega)|^2 S_{qq}(\omega)e^{-i\omega t}d\omega \tag{1.12}$$

Once $S_{qq}(\omega)$ is defined explicitly the integral can be inverted using residue theory, where the poles are those given in (1.5).

Although the addition of three spatial dimensions in the boundary layer problem will add considerable complexity to the analysis, the basic approach is the five steps given

above: (a) the equations governing the boundary layer are derived; (b) the eigenvalues and eigenfunctions of the operators are analyzed which are used to construct a transfer function; (c) the statistical properties of the random free stream turbulence are defined by recourse to experimental observations; (d) the solution is found in real space; and (e) the multi-dimensional statistics of the excited velocity and vorticity field in the boundary layer are obtained. These results can be further distilled to determine such parameters as excited length scales, and the transverse distribution of the excited response in the boundary layer.

In Section 2 the equations governing the flow past the flat plate are derived. The analysis of the eigenvalues and eigenfunctions of the derived operators have been reported in a companion paper (Bridges and Morris, 1985). In Section 3 the free stream turbulence is analyzed and the necessary multidimensional autocorrelation function for the free stream turbulence is specified. In Section 4 the inhomogeneous equations are solved for the velocities, pressure, and vorticities. Fourier transforms are used in the x, z, and t dimensions for obtaining the solution, and the solution in the y-dimension is expressed in terms of a series of Chebyshev polynomials. The y dimension which goes from 0 to ∞ is transformed, for the Chebyshev expansion, to $Y \in [-1,+1]$ using the transformation

$$Y = (y - 2)/(y + 2) \tag{1.13}$$

with metric

$$m(Y) = dY/dy = (1 - Y)^2/4 \tag{1.14}$$

In Section 5 the statistics of the excited field are derived. The multi-dimensional autocorrelation functions are expressed as double Chebyshev expansions, and convenient expressions for the associated matrix of coefficients are constructed.

Residue theory and numerical integration are used to invert the Fourier integrals which result in the correlation functions. The details necessary to complete this are described in Section 6. In Section 7 the correlations are computed for $R = 1000$, 2000,

and 3000 and results are shown of the distribution across the boundary layer of the mean square velocities and vorticities and the Reynolds stress. By evaluating the expressions for various values of the spanwise separation typical spanwise length scales of the excited solution are determined. The main result is the prevalence of three-dimensionality in the solutions. It is shown that if the random free stream turbulence is three-dimensional, then the excited solution will be patently three-dimensional. Comparison with the experimental results of Klebanoff, et al. (1962) shows excellent agreement.

Section 2. The Governing Equations

The equations governing the flow field are the Navier-Stokes equations

$$\nabla \cdot \vec{U} = 0 \tag{2.1}$$

$$\frac{\partial \vec{U}}{\partial t} + (\vec{U} \cdot \nabla)\vec{U} + \frac{1}{\rho} \nabla p - \nu \Delta \vec{U} = 0 \tag{2.2}$$

where

$$\vec{U} = U\hat{i} + V\hat{j} + W\hat{k}$$

Δ is the Laplacian, and the boundary conditions are

$$U = V = W = 0, \quad \text{at} \quad y = 0, \quad \text{for all} \quad z \quad \text{and for} \quad x > 0$$

$$\left.\begin{aligned} U &\to u_\infty + \varepsilon u_s \\ V &\to \varepsilon u_s \\ W &\to \varepsilon w_s \\ P &\to p_\infty + \varepsilon p_s \end{aligned}\right\} \quad \text{as} \quad y \to \infty \tag{2.3}$$

Since the free stream turbulence has amplitude ε, a small parameter, it is proposed that the response in the boundary layer can be expressed as a power series in ε;

$$\begin{aligned} U(x,y,z,t) &= u_0(x,y,z) + \varepsilon u_1(x,y,z,t) + 0(\varepsilon^2) \\ V(x,y,z,t) &= v_0(x,y,z) + \varepsilon v_1(x,y,z,t) + 0(\varepsilon^2) \\ W(x,y,z,t) &= w_0(x,y,z) + \varepsilon w_1(x,y,z,t) + 0(\varepsilon^2) \\ P(x,y,z,t) &= p_0(x,y,z) + \varepsilon p_1(x,y,z,t) + 0(\varepsilon^2) \end{aligned} \tag{2.4}$$

The expansions (2.4) may be substituted into (2.1), (2.2), and (2.3) and terms proportional to like powers of ε, are equated to zero. Only the zeroth and first order problems are considered. The zeroth order problem reduces to the Blasius equation if the usual assumptions of boundary layer theory are used. Then retaining the parallel flow approximation, usual in hydrodynamic stability, the first order problem, after rendering the variables dimensionless using a boundary layer thickness and the free stream velocity, and shifting the dependent variables to render the far field boundary conditions homogeneous, is

$$\Delta p' = -2 \frac{\partial u_0}{\partial y} \frac{\partial}{\partial x} (v' + v_s) \tag{2.5}$$

$$L_4 v' = (1 - u_0) \frac{\partial}{\partial x} \Delta v_s + \frac{\partial^2 u_0}{\partial y^2} \frac{\partial v_s}{\partial x} \tag{2.6}$$

$$L_2 u' = - \frac{\partial u_0}{\partial y} v' - \frac{\partial p'}{\partial x} + (1 - u_0) \frac{\partial u_s}{\partial x} - v_s \frac{\partial u_0}{\partial y} \tag{2.7}$$

$$L_2 w' = - \frac{\partial p'}{\partial z} + (1 - u_0) \frac{\partial w_s}{\partial x} \tag{2.8}$$

where the L_2 and L_4 operators are defined as

$$L_2 \varphi \equiv [\frac{\partial}{\partial t} + u_0 \frac{\partial}{\partial x} - \frac{1}{R} \Delta] \varphi \tag{2.9}$$

$$L_4 \varphi = L_2 \Delta \varphi - \frac{d^2 u_0}{dy^2} \frac{\partial \varphi}{\partial x} \tag{2.10}$$

and the boundary conditions are

$$u'(x,0,z,t) = -u_s(x,0,z,t)$$

$$v'(x,0,z,t) = -v_s(x,0,z,t)$$

$$w'(x,0,z,t) = -w_s(x,0,z,t)$$

$$p'(x,0,z,t) = -p_s(x,0,z,t)$$

$$u',v',w',p' \to 0 \quad \text{as} \quad y \to \infty \tag{2.11}$$

where the primed variables are differences between the first order variables and the far-field boundary values, e.g. $v' = v_1 - v_s$, etc..

The L_4 operator is the well known Orr-Sommerfeld operator. The eigenvalues and eigenfunctions of this operator play an important part in the construction of solutions to the randomly forced problem at hand. A study of the necessary aspects of this operator, as well as the L_2 operator, have been detailed previously by Bridges and Morris (1985).

In the next section the equations for the free stream turbulence are analyzed, and their statistical properties are formulated.

Section 3. Characterization of the Free Stream Turbulence

In this section, aspects of the forcing function, the free stream turbulence, are defined. Using the governing equations for the free stream turbulence, a relationship between the transverse velocity and the other velocities is derived. Therefore only the statistical properties of the transverse velocity need be specified. An autocorrelation function for the transverse velocity is then specified based upon known experimental data gathered in wind tunnels on decaying turbulence. The other statistical properties of the free stream turbulence can be derived from this function.

The equations which govern the free stream turbulence are the first order equations, derived in the last section, evaluated in the free stream as $y \to \infty$. The equations can be solved for a relationship between u_s, v_s, and w_s. Defining transform mates by

$$\hat{v}_s(\alpha;y;\beta,\omega) = \frac{1}{(2\pi)^3} \iiint_{-\infty}^{\infty} v_s(x,y,z,t) e^{-i(\alpha x+\beta z-\omega t)} dx dz dt \tag{3.1}$$

$$v_s(x,y,z,t) = \iiint_{-\infty}^{\infty} \hat{v}_s(\alpha;y;\beta,\omega) e^{i(\alpha x+\beta z-\omega t)} d\alpha d\beta d\omega \tag{3.2}$$

then the other velocity components are,

$$u_s(x,y,z,t) = \frac{\partial}{\partial y} \iiint_{-\infty}^{\infty} \frac{i\alpha}{\alpha^2+\beta^2} \hat{v}_s(\alpha,y,\beta,\omega) e^{i(\alpha x+\beta z-\omega t)} d\alpha d\beta d\omega \tag{3.3}$$

$$w_s(x,y,z,t) = \frac{\partial}{\partial y} \iiint_{-\infty}^{\infty} \frac{i\beta}{\alpha^2+\beta^2} \hat{v}_s(\alpha,y,\beta,\omega) e^{i(\alpha x+\beta z-\omega t)} d\alpha d\beta d\omega \tag{3.4}$$

which may now be used to obtain statistics when the correlation function for v_s is specified.

An autocorrelation function with sufficient generality is chosen to represent the statistical properties of the transverse velocity. As most of the experimental data on the properties of small amplitude free stream turbulence is taken in wind tunnels, the qualitative features of this flow are used as a guide. The autocorrelation function is

chosen to be stationary in time, homogeneous in the transverse and spanwise directions, and inhomogeneous (decaying turbulence) in the streamwise direction. To represent the homogeneous aspects of the correlation function a general Gaussian form is chosen, which includes a specified integral length or time scale. For example, in the transverse direction the correlation function is taken to be

$$R_y(y_1,y_2) = \exp(-\pi(y_2 - y_1)^2/4\ell_y^2) \tag{3.5}$$

where ℓ_y is the integral length of the turbulence in the transverse direction defined by

$$\ell_y = \int_0^\infty \exp\left(-\pi\left(\frac{\eta}{2\ell_y}\right)^2\right)d\eta \tag{3.5a}$$

where $\eta = y_2 - y_1$. This length scale is representative of the characteristics of the largest eddies in the turbulence. In the streamwise direction a decaying Gaussian function is used,

$$R_x(x_1,x_2) = A(x_1)\ \exp\left(-\pi(x_2 - x_1)^2/4\ell_x^2\right) . \tag{3.6}$$

In wind tunnel experiments on decaying turbulence, while the intensity of the turbulence is decaying, the length scale ℓ_x is growing. However, most experiments, for example those of Compte-Bellot and Corrsin (1958), suggest that ℓ_x grows like the square root of the distance downstream, and since it is here rendered dimensionless by a boundary layer thickness, which also grows like the square root of the distance downstream and is considered to be a local constant, in congruence with the parallel flow approximation, ℓ_x can also be considered as a local constant. A plausible expression for the decay of the amplitude of the correlation would be an exponential, $A(x_1) = A_0\exp(-\gamma x_1)$. Later in the calculations, the initial amplitude can be chosen to be unity (essentially included in the small parameter ε), and the results scaled accordingly.

The fact that the turbulence field is superimposed on a uniform velocity field suggests that the proper correlation function should be a convected correlation. Such a convected field is distinguished by the tendency for maximum correlation to lie along a

space-time trajectory where $(x_2 - x_1) - U_c(t_2 - t_1)$ is zero. Consequently, since the turbulence is stationary, and the convection velocity is unity, the streamwise correlation is

$$R_x(x_1,x_2;\tau) = A(x_1)\ \exp\left(-\pi(x_2 - x_1 - \tau)^2/4\ell_x^2\right) . \tag{3.7}$$

Therefore the complete autocorrelation function for the free stream transverse velocity is

$$R_s(x_1,x_2,y_1,y_2,\zeta,\tau) =$$

$$A(x_1)\exp\left[-\frac{\pi}{4}\left(\frac{(x_2 - x_1 - \tau)^2}{\ell_x^2} + \frac{(\eta)^2}{\ell_y^2} + \frac{\zeta^2}{\ell_z^2} + \frac{\tau^2}{\ell_t^2}\right)\right] \tag{3.8}$$

where $\zeta = z_2 - z_1$.

Much of the later analysis will take place under the cloak of Fourier integrals. Therefore the Fourier transform of the x,z,t portion of the autocorrelation function is necessary. Splitting R_s into two parts,

$$R_s(x_1,x_2,y_1,y_2,\zeta,\tau) = R_y(y_1,y_2)Q(x_1,x_2,\zeta,\tau) \tag{3.9}$$

and defining the four dimensional Fourier transform as,

$$\hat{Q}(\alpha_1,\alpha_2,\beta,\omega) = \frac{1}{(2\pi)^4}\iiiint_{-\infty}^{\infty} Q(x_1,x_2\zeta,\tau)e^{-i(\alpha_1x_1+\alpha_2x_2+\beta\zeta-\omega\tau)}dx_1dx_2d\zeta d\tau \tag{3.10}$$

then the desired result is

$$\hat{Q}(\alpha_1,\alpha_2,\beta,\omega) = \frac{A_0\ell_x\ell_z\ell_t}{2\pi^4}\left[\frac{1}{\gamma + i(\alpha_1 + \alpha_2)}\right] \exp[-\theta^2/\pi] \tag{3.11a}$$

where

$$\theta^2 = \ell_x^2\alpha_2^2 + \ell_z^2\beta^2 + (\omega - \alpha_2)^2\ell_t^2 . \tag{3.11b}$$

In the subsequent analysis the solution in the transverse direction will be obtained by finite expansion of the solution in a Chebyshev series. This requires that the forcing function also be expanded in a Chebyshev series,

$$v_s(x,y,z,t) = \sum_{n=0}^{N}{}' q_n(x,z,t)T_n(Y) \tag{3.12}$$

where the prime signifies that the first term is to be halved, and y has been transformed to $Y(y)$, where $Y \in [-1,+1]$. It should be pointed out here that no attempt is made to actually expand the free stream turbulence as a deterministic function. The $q_n(x,z,t)$ are random amplitudes. Consequently, only statistical properties of q_n will be used. Therefore, the accuracy of the Chebyshev representation of the free stream turbulence will rest on how well the autocorrelation of the free stream turbulence can be expanded in a double Chebyshev series; and this can be done quite accurately.

In terms of v_s, the autocorrelation function R_s is

$$R_s(x_1,x_2,y_1,y_2,\zeta,\tau) = \langle v_s^*(x_1,y_1,z,t)v_s(x_2,y_2,z+\zeta,t+\tau)\rangle \tag{3.13}$$

where * denotes the complex conjugate and the brackets denote the ensemble average. Substituting (3.12) into (3.13) results in

$$\langle v_s^*(x_1,y_1,z,t)v_s(x_2,y_2,z+\zeta,t+\tau)\rangle =$$

$$\sum_{n=0}^{N}{}' \sum_{m=0}^{N}{}' \langle q_n^*(x_1,z,t)q_m(x_2,z+\zeta,t+\tau)\rangle\, T_n(Y_1)T_m(Y_2) \tag{3.14}$$

Due to the orthogonality of the Chebyshev polynomials, the auto- and cross-correlations of the q_n are

$$\langle q_n^*(x_1,z,t)q_m(x_2,x+\zeta,t+\tau)\rangle = Q(x_1,x_2,\zeta,\tau)R_{mn} \tag{3.15}$$

where

$$Q(x_1,x_2,\zeta,\tau) = A(x_1)\exp\left[-\frac{\pi}{4}\left(\frac{(x_2-x_1-\tau)^2}{\ell_x^2} + \frac{\zeta^2}{\ell_z^2} + \frac{\tau^2}{\ell_t^2}\right)\right] \tag{3.16}$$

and

$$R_{mn} = \frac{4}{\pi^2}\int_{-1}^{1}\frac{T_n(Y_1)\,dY_1}{\sqrt{1-Y_1^2}}\int_{-1}^{1}\frac{R_y(Y_1,Y_2)T_m(Y_2)\,dY_2}{\sqrt{1-Y_2^2}} \tag{3.17}$$

where

$$R_y(Y_1,Y_2) = \exp\left[-\frac{\pi}{4\ell_y^2}\,[y_2(Y_2)-y_1(Y_1)]^2\right]. \tag{3.18}$$

This bears out the previous conjecture, that the accurate representation of the random transverse velocity in the free stream depends on how accurate the y- autocorrelation function can be expanded in a double Chebyshev series.

As noted, the x, z, t behavior will be analyzed under Fourier integrals. Therefore the necessary analogy of equation (3.15) in the Fourier domain is

$$\langle \hat{q}_n^*(\alpha_1, \omega_1, \beta_1) \hat{q}_m(\alpha_2, \omega_2, \beta_2) \rangle =$$

$$R_{nm} \hat{Q}(-\alpha_1, \alpha_2, \beta_1, \omega_1) \delta(\omega_2 - \omega_1) \delta(\beta_2 - \beta_1) \qquad (3.19)$$

where $\delta(x)$ is the Dirac delta function, R_{nm} is given by (3.17), and $\hat{Q}$ is given by (3.11).

This completes the specification of the statistical properties of the forcing function. It is a function of five parameters; the integral length scales in the x, y, and z directions, the integral time scale, and the decay rate of the turbulence in the streamwise direction. In the next section a method for determining expressions for the excited field in the boundary layer is derived.

Section 4. Solution for the First Order Field

The solution for the first order excited field in the boundary layer follows the analysis of the homogeneous problem outlined in Bridges and Morris (1985). The non-constant coefficients in the equations are function of y only. Therefore a Fourier transform is used in the x, z, and t dimensions. The y dimension, which extends from 0 to ∞, is transformed to the Chebyshev domain, $Y \in [-1,+1]$, using the algebraic transformation,

$$Y = (y - 2)/(y + 2) \tag{4.1}$$

with metric

$$m(Y) = dY/dy = (1 - Y)^2/4 \tag{4.2}$$

The y dependence of the solutions is then expressed in terms of a finite Chebyshev series.

The steps necessary to obtain an expression for the transverse velocity will be gone through in some detail for illustrative purposes and then the solutions for the remainder of the field variables will be only briefly described. Further details can be found in Bridges (1984).

To render the boundary conditions on the *transverse velocity completely homogeneous*, the function v' defined in (2.13) is split into two parts,

$$v_1(x,y,z,t) = v_s(x,y,z,t) - g_1(x,y,z,t) + v(x,y,z,t) \tag{4.3}$$

where

$$g_1(x,y,z,t) = v_s(x,0,z,t)e^{-y} + \left(v_s(x,0,z,t) + \frac{\partial v_s}{\partial y}(x,0,z,t)\right)ye^{-y} \tag{4.4}$$

and v, the only unknown function, is governed by an inhomogeneous partial differential equation with homogeneous boundary conditions. Taking a Fourier transform of this equation in x, z, and t, where the Fourier transform mates are defined in equation (3.2), and multiplying the equation by $-R$ results in the governing equation for v,

$$\hat{L}_4\hat{v} = i\alpha R(u_0 - 1)\left[\frac{d^2}{dy^2} - (\alpha^2 + \beta^2)\right]\hat{v}_s - i\alpha R\,\frac{d^2u_0}{dy^2}\,\hat{v}_s + \hat{L}_4 g_1 \tag{4.5}$$

where

$$\hat{L}_4\hat{\varphi} \equiv \left[iR(\omega - \alpha u_0) + \frac{d^2}{dy^2} - (\alpha^2 + \beta^2)\right]\left[\frac{d^2}{dy^2} - (\alpha^2 + \beta^2)\right]\hat{\varphi} + i\alpha R\,\frac{d^2u_0}{dy^2}\,\hat{\varphi} \tag{4.6}$$

is the well known Orr-Sommerfeld operator. The equation is transformed to the Y domain, using the algebraic transformation (4.1), and the unknown function $\hat{v}$, and the known functions $\hat{v}_s$ and $u_0(Y)$ are expanded in finite Chebyshev series,

$$\hat{v}(\alpha,y,\beta,\omega) = \sum_{n=0}^{N}{}' \hat{v}_n(\alpha,\beta,\omega)T_n(Y) . \tag{4.7}$$

Using the Chebyshev discretization approach described in Bridges and Morris (1985), the equation is integrated, the Chebyshev expansions for $\hat{v}$, $\hat{v}_s$, and u_0 are substituted into the equation, along with the boundary conditions, and the result is a matrix equation,

$$\left[D_4(\alpha,\beta,\omega)\right]\left\{\hat{v}_n\right\} = \left[H_0(\alpha,\beta,\omega)\right]\left\{\hat{q}_n\right\} . \tag{4.8}$$

where the Reynolds number is taken to be specified, and

$$\begin{aligned} D_4(\alpha,\beta,\omega) = {} & C_0(\beta,\omega)\alpha^4 + C_1(\beta,\omega)\alpha^3 + C_2(\beta,\omega)\alpha^2 + \\ & C_3(\beta,\omega)\alpha + C_4(\beta,\omega) \end{aligned} \tag{4.9}$$

is a scalar polynomial with matrix coefficients. This form along with the associated "non-linear in the parameter" eigenvalue problem is discussed in some detail in Bridges and Morris (1984).

By inverting (4.8) the expression for v in real space is given by

$$v(x,y,z,t) = \sum_{n=0}^{N}{}' v_n(x,z,t)\; T_n(Y) \tag{4.10}$$

with the vector $\{v_n\}$ given by

$$\{v_n\} = \iiint_{-\infty}^{\infty} \frac{[C(\alpha,\beta,\omega)][H_0(\alpha,\beta,\omega)]\{\hat{q}_n\}}{\Delta(\alpha,\beta,\omega)}\, e^{i(\alpha x + \beta z - \omega t)} d\alpha d\beta d\omega \tag{4.11}$$

where the matrix C and Δ are the cofactor matrix and determinant respectively, of $D_4(\alpha,\beta,\omega)$.

However, v is only part of v_1. The Chebyshev coefficient vector for v_s is given by

$$\{q_n\} = \iiint_{-\infty}^{\infty} \{\hat{q}_n\}\, e^{i(\alpha x + \beta z - \omega t)} d\alpha d\beta d\omega \tag{4.12}$$

and the boundary condition function g_1 is also expanded in a Chebyshev series,

$$g_1(x,y,z,t) = \sum_{n=0}^{N} g_{1n}(x,z,t)\, T_n(Y) \tag{4.13}$$

with vector

$$\{g_{1n}\} = \iiint_{-\infty}^{\infty} [G_1]\{\hat{q}_n\} e^{i(\alpha x + \beta z - \omega t)} d\alpha d\beta d\omega \tag{4.14}$$

where the matrix G_1 is given in Appendix A. Combining these terms results in the Chebyshev expansion for v_1,

$$v_1(x,y,z,t) = \sum_{n=0}^{N}{}' \, v_{1n}(x,z,t)\, T_n(Y) \tag{4.15}$$

with the coefficient vector given by,

$$\{v_{1n}\} = \iiint_{-\infty}^{\infty} \left[\frac{[C(\alpha,\beta,\omega)][H_0(\alpha,\beta,\omega)]}{\Delta(\alpha,\beta,\omega)} + [I - G_1]\right]\{\hat{q}_n\} e^{i(\alpha x+\beta z-\omega t)} d\alpha d\beta d\omega \tag{4.16}$$

If the expression for $\{\hat{q}_n\}$ were known, the integral (4.16) could be inverted using residue theory where the poles would arise from $\Delta(\alpha,\beta,\omega) = 0$, and from any poles of $\{\hat{q}_n\}$. In a subsequent section it will be shown that the poles of $\Delta(\alpha,\beta,\omega)$ are exponentially larger than the poles due to the forcing function. Therefore, in the calculations the contribution of the matrix
$[I - G_1]$ to the overall solution is negligible.

The pressure $p'(x,y,z,t)$ satisfies the Poisson equation given in (2.14). Using the y-momentum equation, it can be shown that

$$\hat{p}' = \frac{1}{\alpha^2 + \beta^2} \{\frac{1}{R} \frac{d^3\hat{v}'}{dy^3} + [i(\omega - \alpha u_0) - \frac{1}{R} (\alpha^2 + \beta^2)]\frac{d\hat{v}'}{dy} + i\alpha \frac{du_0}{dy} \hat{v}'\} +$$

$$\frac{1}{\alpha^2 + \beta^2} \{i\alpha(1 - u_0) \frac{d\hat{v}_s}{dy} + i\alpha \frac{d\hat{u}_0}{dy} \hat{v}_s\} \tag{4.17}$$

where $v'(x,y,z,t) = v(x,y,z,t) - g_1(x,y,z,t)$. The solution for p_1 in real space is then

$$p_1(x,y,z,t) = p_s(x,y,z,t) + \iiint_{-\infty}^{\infty} \hat{p}'(\alpha,y,\beta,\omega)e^{i(\alpha x+\beta z-\omega t)}d\alpha d\beta d\omega \ . \tag{4.18}$$

The governing equation for the streamwise velocity $u'(x,y,z,t)$ is given by equation (2.16). If a boundary condition function is used to render the boundary conditions homogeneous, then u_1 has the expression

$$u_1(x,y,z,t) = u(x,y,z,t) + u_s(x,y,z,t) - g_2(x,y,z,t) \tag{4.19}$$

with $g_2 = e^{-y} u_s(x,0,z,t)$, and the only unknown is $u(x,y,z,t)$. After substitution of (4.19), and subsequent exercise of the Fourier transform in $x,z,$ and $t,$ the governing equation for $\hat{u}$ is

$$\hat{L}_2\hat{u} = -i\alpha R(1 - u_0)\hat{u}_s + R \frac{du_0}{dy}(\hat{v}' + \hat{v}_s) + i\alpha R\hat{p}' + \hat{L}_2\hat{g}_2 \tag{4.20}$$

where

$$\hat{L}_2\hat{u} = \frac{d^2\hat{u}}{dy^2} + [iR(\omega - \alpha u_0) - (\alpha^2 + \beta^2)]\hat{u} \ . \tag{4.21}$$

However $\hat{u}_s$ is related to $\hat{v}_s$ by

$$\hat{u}_s = \frac{i\alpha}{\alpha^2 + \beta^2} \frac{d\hat{v}_s}{dy} \tag{4.22}$$

and $\hat{p}'$ can be expressed in terms of $\hat{v}$ and $\hat{v}_s$. Then (4.20) becomes

$$\hat{L}_2\hat{u} = \hat{L}_2\hat{g}_2 + \frac{\beta^2 R}{\alpha^2 + \beta^2} \frac{du_0}{dy} \hat{v}_s + \frac{\beta^2 R}{\alpha^2 + \beta^2} \frac{du_0}{dy} \hat{v}'$$

$$+ \frac{i\alpha}{\alpha^2 + \beta^2} \{\frac{d^3\hat{v}'}{dy^3} + [iR(\omega - \alpha u_0) - (\alpha^2 + \beta^2)]\frac{d\hat{v}'}{dy}\} \tag{4.23}$$

the unknown $\hat{u}$ is then expanded in a finite Chebyshev series,

$$\hat{u}(\alpha;y;\beta,\omega) = \sum_{N=0}^{N}{}' \hat{u}_n(\alpha,\beta,\omega)\ T_n(Y) \tag{4.24}$$

Substituting (4.24) as well as the Chebyshev series for the free stream velocity, and the known transverse velocity, the following matrix equation results,

$$[D_2(\alpha,\beta,\omega)]\{\hat{u}_n\} = [H_1(\alpha,\beta,\omega)]\{\hat{v}_n\} + [H_3(\alpha,\beta,\omega)]\{\hat{q}_n\} \tag{4.25}$$

where $\{\hat{v}_n\}$ is given by (4.8). The poles of $D_2(\alpha,\beta,\omega)$ are all quite stable (a proof of this is given in Appendix C) and will lend little to the overall solution by comparison to the poles of $\{\hat{v}_n\}$. Therefore,

$$\{\hat{u}_n\} = \frac{[D_2^{-1}(\alpha,\beta,\omega)][H_1(\alpha,\beta,\omega)][C(\alpha,\beta,\omega)][H_0(\alpha,\beta,\omega)]}{\Delta(\alpha,\beta,\omega)}\{\hat{q}_n\}\ . \tag{4.26}$$

Taking into account the expressions for u_s and g_2, the first order streamwise velocity has a Chebyshev expansion,

$$u_1(x,y,z,t) = \sum_{N=0}^{N}{}' u_{1n}(x,z,t)\ T_n(Y) \tag{4.27}$$

with the vector $\{u_{1n}\}$ given by

$$\{u_{1n}\} = \iiint_{-\infty}^{\infty}\Big[\frac{[D_2^{-1}][H_1][C][H_0]}{\Delta(\alpha,\beta,\omega)} +$$

$$\frac{i\alpha}{\alpha^2+\beta^2}[I - G_2][D_1]\Big]\{\hat{q}_n\}e^{i(\alpha x+\beta z-\omega t)}d\alpha d\beta d\omega \tag{4.28}$$

and the matrices G_2 and D_1 are described in Appendix A.

Using a similar procedure, the complete expression for w_1, the first order spanwise velocity is

$$w_1(x,y,z,t) = \sum_{N=0}^{N}{}' w_{1n}(x,z,t)\ T_n(Y) \tag{4.29}$$

with the associated vector given by

$$\{w_{1n}\} = \iiint_{-\infty}^{\infty}[\frac{i\beta[D_2^{-1}][H_2][C][H_0]}{\Delta(\alpha,\beta,\omega)} + \frac{i\beta}{\alpha^2+\beta^2}[I - G_2][D_1]]\{\hat{q}_n\} \cdot$$

$$e^{i(\alpha x+\beta z-\omega t)}d\alpha d\beta d\omega \tag{4.30}$$

The excited first order vorticity field in the boundary layer is also of considerable interest. The streamwise vorticity is given here. The other two components can be derived in a similar manner. The first order streamwise vorticity is defined by

$$\Omega_1 = \frac{\partial w_1}{\partial y} - \frac{\partial v_1}{\partial z} \, . \tag{4.31}$$

Operating on w_1 and v_1 results in the desired expression for Ω_1,

$$\Omega_1(x,y,z,t) = \sum_{N=0}^{N}{}' \, \Omega_{1n}(x,z,t) \; T_n(Y) \tag{4.32}$$

with

$$\{\Omega_{1n}\} = \iiint_{-\infty}^{\infty} i\beta[[[D_1][D_2^{-1}][H_2] - [I]] \frac{[C][H_0]}{\Delta(\alpha,\beta,\omega)} + I - G_1 +$$

$$\frac{[D_1](I-G_2][D_1]}{\alpha^2+\beta^2}]\{\hat{q}_n\}e^{i(\alpha x+\beta z-\omega t)}d\alpha d\beta d\omega \tag{4.33}$$

where D_1, G_1, and G_2 are defined in the Appendix A, and D_2, H_0, C, and H_2, defined previously, are all functions of α, β, and ω. This expression is not, however, as complicated as it looks. The residues from the boundary terms are negligible in comparison to the residues from $\Delta(\alpha,\beta,\omega) = 0$ so that these terms may be neglected. Furthermore, since $[C][H_0]/\Delta$ is the solution matrix for $v_1(x,y,z,t)$, the above equation involves merely a pre-multiplication of the v_1 result.

This completes the solution for the excited first orderfield in the boundary layer expressed in terms of physical variables. In the next section the statistics of this result are derived.

Section 5. Statistics of the First Order Solution

The main concern, in the present analysis of the effect of free stream disturbances on the laminar boundary layer, is the situation where the free stream disturbances are random. The vector $\{q_n\}$, the Chebyshev coefficients of the free stream transverse velocity, is a random vector. The complex outer product of this vector results in a matrix of terms which include the known statistical properties of the free stream turbulence.

The solutions for the first order velocity, pressure, and vorticity field, derived in the previous section, were shown to be expressed in terms of the vector $\{q_n\}$. To arrive at the statistical properties of the resulting field, the complex outer product, of the Chebyshev vectors of the various dependent variables, is taken, resulting in a relationship between the statistical response in the boundary layer and the statistics of the free stream turbulence.

In what follows the multidimensional two-point autocorrelation functions for the three velocity components are derived. The derivation of the statistics of the transverse velocity is given in some detail, and the results for the remainder of the field follows in a similar manner, and therefore the results only are given. The cross-correlations for the field variables can also be derived in a similar manner, and this is straightforward. The autocorrelation for the streamwise vorticity is also given. The other vorticity correlations can be derived in a similar manner.

The solution for the transverse velocity, v_1, has the Chebyshev expansion,

$$v_1(x,y,z,t) = \sum_{N=0}^{N}{}' \, v_{1n}(x,z,t)\, T_n(Y) \tag{5.1}$$

The two-point autocorrelation of this function is simply $\langle v_1^*, v_1\rangle$. In terms of the Chebyshev series this becomes

$$\begin{aligned}\langle v_1^*, v_1\rangle &= \langle v_1^*(x_1,y_1,z,t), v_1(x_2,y_2,z+\zeta,t+\tau)\rangle \\ &= \sum_{n=0}^{N}{}' \sum_{m=0}^{N}{}' \langle v_{1m}^*(x_1,z,t), v_{1n}(x_2,z+\zeta,t+\tau)\rangle T_n(Y_1) T_m(Y_2)\end{aligned} \tag{5.2}$$

The autocorrelation function is therefore a double Chebyshev series, the coefficients of which form a matrix with $(N+1)^2$ entries. This matrix is constructed by taking the complex outer product of the vector $\{v_{1n}\}$. This matrix will be denoted by a capital letter, i.e. $[V_1]$. Thus $[V_1]$ may be expressed as

$$[V_1] = \{v_{1n}^*\} \cdot \{v_{1n}\}^T \tag{5.3}$$

where the superscript T denotes transpose. The vector $\{v_{1n}\}$ is given in equation (4.16). Before proceeding further, it is important to notice the following. The matrices $[I]$ and $[G_1]$ in the integrand are exponentially smaller than the main term, as will be shown in the next section, and may therefore be neglected. Then substitution of (4.16) into (5.3) results in

$$[V_1] = \int\!\!\int\!\!\int\!\!\int\!\!\int\!\!\int_{-\infty}^{\infty} \frac{[C^*(\alpha_1,\beta_1,\omega_2)][H_0^*(\alpha_1,\beta_1,\omega_1)]\{q_n^*\}\cdot\{q_m\}^T[H_0(\alpha_2,\beta_2,\omega_2)]^T[C(\alpha_2,\beta_2,\omega_2)]^T}{\Delta^*(\alpha_1,\beta_1,\omega_1)\Delta(\alpha_2,\beta_2,\omega_2)} \cdot$$

$$e^{i[-\alpha_1 x_1-\beta_1 z+\omega_1 t+\alpha_2 x_2+\beta_2(z+\zeta)-\omega_2(t+\tau)]}\, d\alpha_1 d\alpha_2 d\beta_1 d\beta_2 d\omega_1 d\omega_2 \tag{5.4}$$

However, it was shown in Section 3 that

$$\{\hat{q}_n^*\}\cdot\{q_m\}^T = [R]\hat{Q}(-\alpha_1,\alpha_2,\beta_1,\omega_1)\delta(\omega_2-\omega_1)\delta(\beta_2-\beta_1) \tag{5.5}$$

where the entries of the matrix [R] are defined in equation (3.17), and the scalar valued function $\hat{Q}$ is defined in equation (3.11). The Orr-Sommerfeld operator also contains certain useful relationships between itself and its complex conjugate,

$$D_4^*(\alpha,\beta,\omega) = D_4(-\alpha,\beta,-\omega) \tag{5.6a}$$

$$D_4(\alpha,-\beta,\omega) = D_4(\alpha,\beta,\omega) \tag{5.6b}$$

$$\alpha(\beta,-\omega) = -\alpha^*(\beta,\omega) \tag{5.6c}$$

Proof of these relations can be found in the appendix of the paper by Tam and Chen (1976). In the same manner, it can be shown that

$$H_0^*(\alpha,\beta,\omega) = H_0(-\alpha,\beta,\omega) \tag{5.6d}$$

Substituting (5.5) and (5.6) into (5.4) results in

$$[V_1] = \iiiint_{-\infty}^{\infty} \frac{[C(\alpha_1,\beta,-\omega)]\,[H_0(\alpha_1,\beta,-\omega)]\,[R]\,[H_0(\alpha_2,\beta,\omega)]^T [C(\alpha_2,\beta,\omega)]^T}{\Delta(\alpha_1,\beta,-\omega)\Delta(\alpha_2,\beta,\omega)} \cdot$$

$$\hat{Q}(\alpha_1,\alpha_2,\beta,\omega)e^{i(\alpha_1 x_1+\alpha_2 x_2+\beta\zeta-\omega\tau)} d\alpha_1 d\alpha_2 d\beta d\omega \tag{5.7}$$

This integral can be inverted using residue theory. Computational aspects of this procedure are discussed in the next section.

Since the procedure is similar for the other correlations, only the necessary details are repeated in the other derivations.

The two-point autocorrelation for u_1 has the expression

$$\langle u_1^*, u_1\rangle = \sum_{n=0}^{N}{}' \sum_{m=0}^{N}{}' \langle u_{1n}{}^*(x_1,z,t), u_{1m}(x_2,z+\zeta,t+\tau)\rangle T_n(Y_1)T_m(Y_2) \tag{5.8}$$

The necessary matrix of coefficients is given by the complex outer product of the vector $\{u_{1n}\}$ given in equation (4.28). Neglecting the higher order terms the result for $[U_1]$ is

$$[U_1] = \iiiint_{-\infty}^{\infty} [D_2^{-1}(\alpha_1,\beta,-\omega)]\,[H_1(\alpha_1,\beta,-\omega)]\,[\hat{V}_1(\alpha_1,\alpha_2,\beta,\omega)]\,[H_1(\alpha_2,\beta,\omega)]^T \cdot$$

$$[D_2^{-1}(\alpha_2,\beta,\omega)]^T \cdot \hat{Q}(\alpha_1,\alpha_2,\beta,\omega)e^{i(\alpha_1 x_1+\alpha_2 x_2+\beta\zeta-\omega\tau)} d\alpha_1 d\alpha_2 d\beta d\omega \tag{5.9}$$

where

$$[\hat{V}_1(\alpha_1,\alpha_2,\beta,\omega)] =$$

$$\frac{[C(\alpha_1,\beta,-\omega)]\,[H_0(\alpha_1,\beta,-\omega)]\,[R]\,[H_0(\alpha_2,\beta,\omega)]^T [C(\alpha_2,\beta,\omega)]^T}{\Delta(\alpha_1,\beta,-\omega)\Delta(\alpha_2,\beta,\omega)} \tag{5.10}$$

The two-point autocorrelation for w_1 has the expression

$$\langle w_1{}^*, w_1\rangle = \sum_{n=0}^{N}{}' \sum_{m=0}^{N}{}' \langle w_{1n}{}^*(x_1,z,t), w_{1m}(x_2,z+\zeta,t+\tau)\rangle T_n(Y_1)T_m(Y_2) \tag{5.11}$$

with the necessary matrix of coefficients, $[W_1]$, given by the complex outer product of the vector $\{w_{1n}\}$, given in equation (4.30). Neglecting the higher order terms, the resulting expression for $[W_1]$ is

$$[W_1] = \iiiint_{-\infty}^{\infty} \beta^2 [D_2^{-1}(\alpha_1,\beta,-\omega)][H_2(\alpha_1,\beta,-\omega)][\hat{V}_1(\alpha_1,\alpha_2,\beta,\omega)][H_2(\alpha_2,\beta,\omega)]^T$$

$$[D_2^{-1}(\alpha_2,\beta,\omega)]^T \cdot \hat{Q}(\alpha_1,\alpha_2,\beta,\omega) e^{i(\alpha_1 x_1+\alpha_2 x_2+\beta\zeta-\omega\tau)} d\alpha_1 d\alpha_2 d\beta d\omega \tag{5.12}$$

where $[\hat{V}_1]$ is given in (5.10).

The vorticity correlations can also be derived. For example, the streamwise vorticity, which was derived in the previous section, has the following two-point correlation,

$$\langle \Omega_1^*, \Omega_1\rangle = \sum_{n=0}^{N}{}' \sum_{m=0}^{N}{}' \langle \Omega_{1n}{}^*(x_1,z,t), \Omega_{1m}(x_2,z+\zeta,t+\tau)\rangle T_n(Y_1)T_m(Y_2) \tag{5.13}$$

Defining the associated matrix of coefficients by $[\Lambda_1]$, after neglecting the higher order terms, the necessary matrix of coefficients is

$$[\Lambda_1] = \iiiint_{-\infty}^{\infty} \beta^2 [[D_1][D_2^{-1}(\alpha_1,\beta,-\omega)][H_2(\alpha_1,\beta,-\omega)]-[I]][\hat{V}_1(\alpha_1,\alpha_2,\beta,\omega)] \cdot$$

$$[[D_1][D_2^{-1}(\alpha_2,\beta,\omega)][H_2(\alpha_2,\beta,\omega)]-[I]]^T \hat{Q}(\alpha_1,\alpha_2,\beta,\omega) \cdot$$

$$e^{i(\alpha_1 x_1+\alpha_2 x_2+\beta\zeta-\omega\tau)} d\alpha_1 d\alpha_2 d\beta d\omega \tag{5.14}$$

where $[\hat{V}_1]$ is defined in equation (5.10).

These integrals can all be evaluated in a similar manner using residue theory in conjunction with numerical integration. The calculation centers on the computation of

$[V_1]$, with the other evaluations requiring a pre- and/or post-multiplication of this result. The computational details needed to complete the procedure are given in the next section.

Section 6. Inversion of the Fourier Integrals

The integrals in the previous section are of the type

$$\Gamma(x,z,t) = \iiint_{-\infty}^{\infty} \frac{\hat{F}(\alpha,\beta,\omega)}{\Delta(\alpha,\beta,\omega)} e^{i(\alpha x+\beta z-\omega t)} d\alpha d\beta d\omega \tag{6.1}$$

where $F(\alpha,\beta,\omega)$ is an entire function, and $\Delta(\alpha,\beta,\omega)$ is a determinant expressing the relationship between α and ω for particular values of β, often referred to as the dispersion relationship. Although a Fourier transform has been used in time in the analysis, and a Fourier transform is to be used in the eventual computation, the formalism for inverting the transforms is derived using a Laplace transform in time. The formalism can be found in Chapter 2 of the monograph by Briggs (1964). Neglecting β for brevity, the expression (6.1) after using residue theory is

$$F(x,t) = 2\pi i \sum_{k=1}^{M} \int_{-\infty+i\sigma}^{\infty+i\sigma} \frac{\hat{F}(\alpha_k,\omega)}{\left.\frac{\partial}{\partial\alpha}\Delta(\alpha,\omega)\right|_{\alpha=\alpha_k}} e^{i(\alpha_k x-\omega t)} d\omega \tag{6.2}$$

The ω-integration is deformed down to the real axis while avoiding all real values of α which produce complex values of ω with positive imaginary parts. In addition in the sum in (6.2) only the first term will be retained as it produces the dominant contribution. A typical spectrum for the determinant is shown in Figures (1a) and (1b). There is an isolated unstable mode and then a set of discrete approximations to the continuous spectrum which contribute very little to the solution.

Reinstating β, the integral (6.1) can now be evaluated further. First the ranges of integration for ω and β may be simplified. By virtue of equations (5.6), the integrands are even functions of β. Using the $[V_1]$ matrix as an example, the result is

$$[V_1] = 2\int_0^{\infty} \iiint_{-\infty}^{\infty} I(\alpha_1,\alpha_2,\beta,\omega) d\alpha_1 d\alpha_2 d\omega d\beta \tag{6.3}$$

where

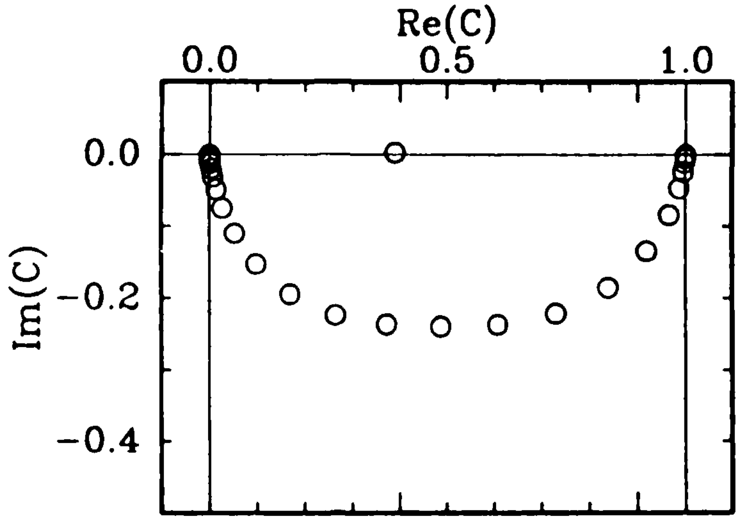

Figure 1a. The eigenvalue spectrum for the Blasius boundary layer for the case of spatial stability plotted in C-space for $R = 600$, $\omega = 0.12$, $\beta = 0$, and $N = 36$.

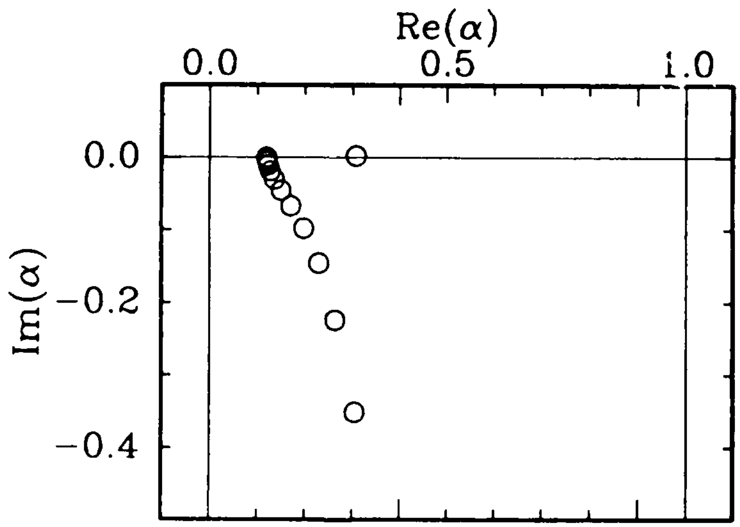

Figure 1b. The eigenvalue spectrum for the Blasius boundary layer plotted in α-space for $\omega = 0.12$, $\beta = 0.$, $R = 600$, and $N = 36$.

$$I(\alpha_1,\alpha_2,\beta,\omega) = \frac{[C(\alpha_1,\beta,-\omega][H_0(\alpha_1,\beta,-\omega)][H_0(\alpha_2,\beta,\omega)]^T[C(\alpha_2,\beta,\omega)]^T}{\Delta(\alpha_1,\beta,-\omega)\Delta(\alpha_2,\beta,\omega)} \cdot$$

$$\hat{Q}(\alpha_1,\alpha_2,\beta,\omega)e^{i(\alpha_1+\alpha_2)x} \tag{6.4}$$

but splitting the ω integral into an integral from $-\infty$ to 0 plus an integral from 0 to ∞, and noting that

$$I(\alpha_1,\alpha_2,\beta,-\omega) = I^*(-\alpha_1,-\alpha_2,\beta,\omega) \tag{6.5}$$

the integral becomes

$$[V_1] = 2\int\!\!\int_0^{\infty}\int\!\!\int_{-\infty}^{\infty} [I(\alpha_1,\alpha_2,\beta,\omega)+I^*(-\alpha_1,-\alpha_2,\beta,\omega)]d\alpha_1 d\alpha_2 d\beta d\omega \tag{6.6}$$

The result (6.6) is proved using the relations given in equations (5.6). The other integrals can be treated similarily.

As suggested in the last section, the Fourier integrals are inverted using residue theory in α_1 and α_2, while the ω and β integrations are carried out using numerical integration. It was also suggested that for each value of ω and β, there is but one residue in each of the α-planes which need be evaluated. It is the purpose here to see what other poles are present, illustrating the complete solution possibilities, and where and why many of these poles can be neglected. The form for the solution can be expressed generically as

$$\varphi(x) = \int_{\Omega} K(\Omega,X)\ f(\Omega)\ d\Omega \tag{6.7}$$

where φ can be a correlation function for any of the sought field variables, and X, representing spatial and temporal separation, is the solution space. $f(\Omega)$ is a spectral function representative of the random forcing function, defined in a spectral volume Ω. $K(\Omega,X)$ is a deterministic transfer function which represents the operator. This is the classical form for a random forced vibration problem. There are obviously two sources for poles in the integrand, the transfer function K, and the forcing function f.

To illustrate these ideas it is necessary to examine only the expression for the v_1 autocorrelation and the u_1 autocorrelation. The others are derived from these or have the same poles.

Equation (5.7) of the last section defines the matrix necessary for the v_1 correlation. The matrix [R] and the function Q are representative of the forcing function. Because the expression is in terms of matrices, the matrix [R] is sandwiched in the midst of the transfer function. The transfer function is represented by the remaining matrices, [C] and $[H_0]$, and the determinant, which appears in the denominator.

The forcing function has one pole. If the integration over α_2 is performed first, there is a pole at $\alpha_2 = i\gamma - \alpha_1$, as can be seen in equation (3.11). However, when this pole is substituted into the exponential terms, $\exp(i\alpha_2 x_2)$, the result is $\exp(-\gamma x_2 - i\alpha_1 x_2)$. Since $\gamma > 0$, and x_2 is proportional to the Reynolds number, this term is exponentially small in comparison to contributory poles with positive exponentials. Therefore, this pole can be safely excluded from the calculation. If this pole had a more significant contribution it would result in the forcing function imposing its character on the boundary layer. This is a phenomenon which is often noticed in experiments with large amplitude free stream tubulence. Reshotko (1976) refers to this phenomenon as "high-intensity bypass". It is possible that, for the case of strong free stream turbulence, this pole dominates the other poles.

In the solution for the v_1 autocorrelation matrix, the remaining contributing poles are due to the roots of $\Delta(\alpha,\beta,\omega) = 0$. These are the natural modes of the Orr-Sommerfeld operator, discussed in some detail by Bridges and Morris (1985). There will be both upstream traveling modes $(Re(\alpha) < 0)$, and downstream traveling modes $(Re(\alpha) > 0)$. Numerical evidence suggests that the modes with $Re(\alpha) < 0$ form a continuous spectrum and are all damped. Whether there are discrete upstream modes or unstable modes is unknown. The remaining spectrum can be broken up, for discussion purposes, into three parts; (a) the single dominant mode, (b) the higher modes, and (c) the continuous spectrum (or

discrete approximation of it). All three contribute poles to the solution. However, in the range of integration, there is a band of frequencies and spanwise wavenumbers which produce eigenvalues with negative imaginary parts. The other poles always have negative exponentials. Therefore, the single dominant mode produces a significantly greater contribution, and the other poles can be safely neglected. A similar discussion can be put forward for the u_1 auto-correlation. The matrix necessry for this correlation is given in equation (5.9) and (5.10). Since the v_1 autocorrelation matrix appears in this expression, it will have the poles discussed above, with the same argument holding for retention of the dominant mode only. However, in this expression, there are two additional sources for poles. The operator matrix $D_2(\alpha,\beta,\omega)$ has poles. These, however, can be safely neglected. In Appendix C it is proved that the poles of this operator are heavily damped and offer little to the solution. The other source is the matrix $H_1(\alpha,\beta,\omega)$. This matrix is a discretization of the right hand side of equation (4.23). Note that there is a term $\alpha^2 + \beta^2$, in the denominator. This gives rise to poles at $\alpha = \pm i\beta$. Since there is no branch cut in the α-plane between $\alpha = -i\beta$ and $\alpha = +i\beta$, the contour of integration is deformed such that only the pole at $\alpha = +i\beta$ will be inside of the contour. But this pole produces an exponential weighting of the form $\exp(-\beta x)$, again an exponentially small term.

In conclusion, it is clear that the forcing function and operator, or transfer function, are rich in their possibilities for solution, but the exponential term in the integrand filters out many of these possibilities leaving the dominant mode as the residuum.

With these ideas in mind the Fourier transforms can now be inverted. Using the $[v_1]$ autocorrelation matrix for illustration and setting $\zeta = \tau = 0$ for brevity, the result after integration over α_1 and α_2 is

$$[V_1] = -8\tau^2 \iint_0^\infty [I(\beta,\omega) + I^*(\beta,\omega)]\,d\omega\,d\beta \qquad (6.8)$$

where

$$I(\beta,\omega) = \frac{[C(-\alpha_k^*,\beta,-\omega)]\,[H_0(-\alpha_k^*,\beta,-\omega)]\,[R]\,[H_0(\alpha_k,\beta,\omega)]^T[C(\alpha_k,\beta,\omega)]^T}{\Delta^{(1)}(-\alpha_k^*,\beta,-\omega)\Delta^{(1)}(\alpha_k,\beta,\omega)} \cdot$$

$$\hat{Q}(-\alpha_k^*,\alpha_k,\beta,\omega)e^{i(\alpha_k-\alpha_k^*)x} \tag{6.9}$$

and α_k satisfies $\Delta(\alpha_k,\beta,\omega) = 0$, and

$$Q(-\alpha_k^*,\alpha_k,\beta,\omega) = \frac{A_0\ell_x\ell_z\ell_t}{2\pi^4}\left[\frac{1}{\gamma + i(\alpha_k-\alpha_k^*)}\right]e^{-\theta^2/k} \tag{6.10a}$$

where

$$\theta^2 = \ell_x{}^2\alpha_k{}^2 + \ell_z{}^2\beta^2 + (\omega - \alpha_k)^2\ell_t{}^2 \tag{6.10b}$$

and

$$\Delta^{(1)} \equiv d\Delta/d\alpha \ .$$

Before proceeding further, it is obvious that it will be necessary to evaluate the derivative of a determinant at a singular point; $\Delta(\alpha_k,\beta,\omega) = 0$, where it is assumed that α_k is a simple root.

A general efficient method for evaluating the local derivative of a determinant at singular and non-singular points, as well as higher derivatives, has recently been developed by Bridges and Vaserstein (1985). A brief outline of the necessary details are presented in Appendix B.

After the residue theory is used in the integral over α_1 and α_2, the remaining work is a numerical integration of an integral of the form

$$F = \int_0^{\bar{\beta}}\int_0^{\bar{\omega}} f(\beta,\omega)d\omega d\beta \tag{6.11}$$

and this is carried out, in the classical manner, using Simpson's rule.

This completes the computational details needed to evaluate explicitly the statistical properties of the excited solution in the boundary layer. In the next section results of the evaluations are presented.

Section 7. Results

Since the zeroth order solution is the well known Blasius solution, the present study need only concentrate on the perturbation to this; the statistical results for the first order field. The set of parameters governing the first order set of solutions is numerous. Representative of the free stream turbulence are the length scales ℓ_x, ℓ_y, and ℓ_z, the time scale, ℓ_t, and the decay rate γ. The statistical response can be evaluated in a variety of ways. The response consists of auto- and cross-correlations (or the spectra) of the three velocity components, the pressure, and the three components of vorticity. As the correlations and spectra are interchangeable, only the correlation results are considered. Each of the correlation functions are dependent upon the separation distances in x, y, z, and t.

There is also numerical accuracy to consider. The numerical considerations are (a) the order of the series expansions in terms of Chebyshev polynomials, and (b) the numerical integration in $\omega - \beta$ space. In all the results presented, N, the order of the Chebyshev expansion is greater than or equal to 24. Numerical tests with other values of N have shown this to be sufficient. For the numerical integration of the spectral integrands in $\omega - \beta$ space, after testing several nets, a value of 11 points in ω and 9 points in β was chosen.

Rather than concentrate on a parametric study, the main theme will be to elucidate the nature of the three dimensionality in the response of the boundary layer. For the set of results presented, the following parmeters are used for the free stream turbulence; $\ell_x = 1/5$, $\ell_y = 1/5$, $\ell_z = 1/5$, $\ell_t = 1/5$, and $\gamma = .01$, where these parameters have been normalized by the Blasius boundary layer thickness and the free stream velocity.

The subscripts for the first order solutions will be dropped for brevity. The first order y-velocity will be v, etc.

Figures 2, 3, and 4 show the transverse distributions of the excited response at R = 1000 (R here is based on displacement thickness). In Figure 2, the rms distribution for

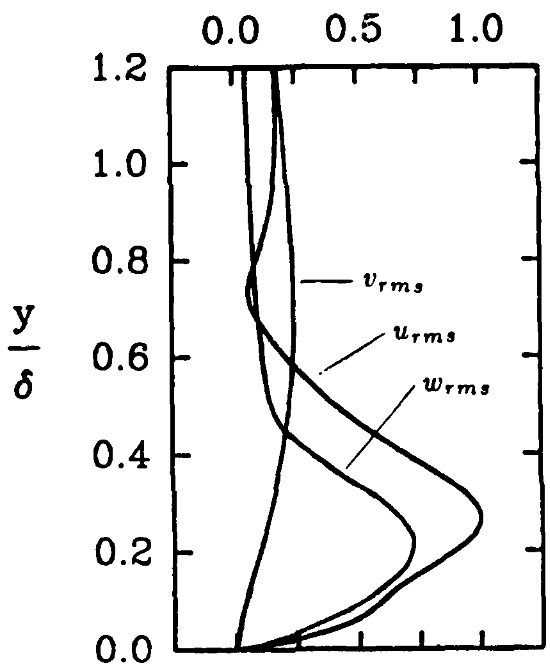

Figure 2. The distribution across the boundary layer of the rms values of the perturbation velocity field for $R = 1000$.

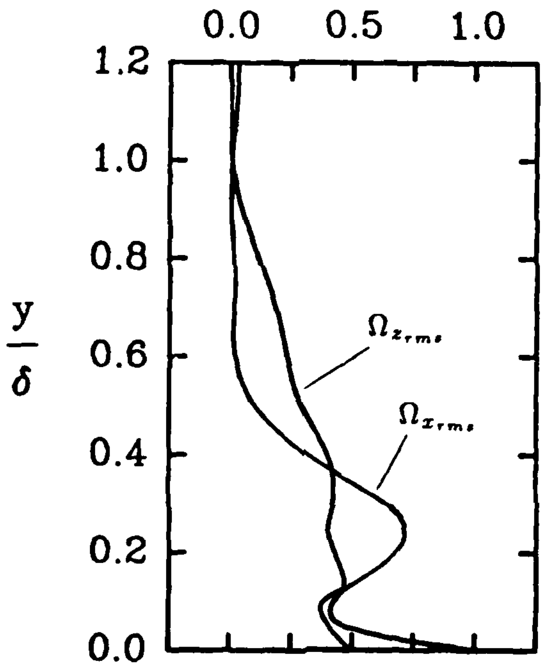

Figure 3. The distribution across the boundary layer of the rms value of the perturbation vorticity field for $R = 1000$.

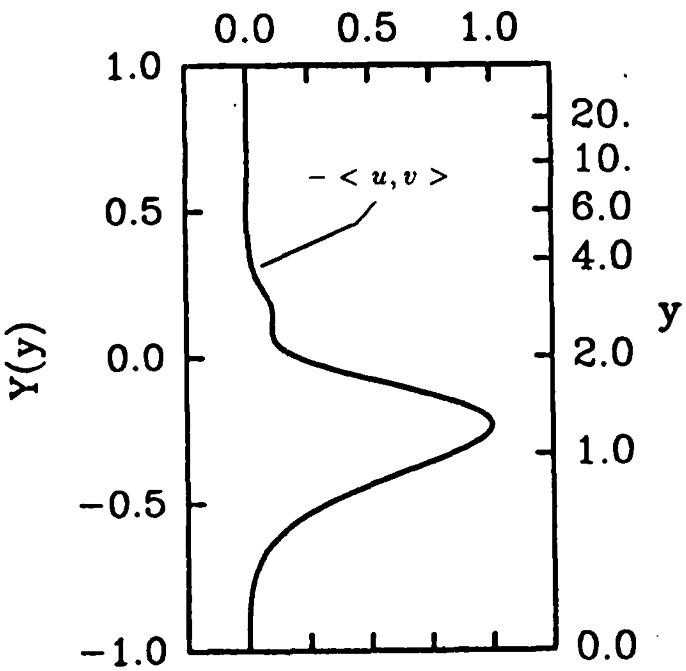

Figure 4. The transverse distribution of the Reynolds stress at zero delay for $R = 1000$.

the three velocity components is plotted (this is the square root of the autocorrelation function at zero delay) versus the distance across the boundary layer ($y/\delta = 1$ is at the edge of the Blasius boundary layer). The most important feature here is that all three components are of a similar magnitude, emphasizing the three-dimensionality of the response. The magnitudes have been scaled such that the maximum of the streamwise component is unity with the other components scaled accordingly. The actual magnitude vary widely as the parameters governing the free stream turbulence are varied (the excited magnitudes range from order 10 to order 1000). However, the qualitative features of the excited field are invariant to the changes in magnitude. Therefore in these results the emphasis will be on the qualitative features and comment on the relationship between the free stream parameters and the excited magnitudes must be reserved until a more complete parametric study can be performed.

It is also interesting to note the hump in the distribution of u_{rms} near the wall. This is a distinctive feature of the exerimental results of Klebanoff, et. al (1962) and will be discussed further at the end of this section. The maximum of each of the functions occurs near the wall with the exception of V_{rms}. This in fact holds true for other Reynolds numbers (up to 3000) as well. The maximum of V_{rms} occurs at about 60% of the boundary layer thickness. According to Klebanoff, et. al. (1962) this is the region where non-linear breakdown occurs. It is likely that an instantaneous shear layer is set up in this region due to the fact that $\partial v/\partial y$, on average, is zero there, which is highly unstable.

In Figure 2 the relative magnitudes are $(v_{rms})_{max} = .38\ (u_{rms})_{max}$ and $(w_{rms})_{max} = .77(u_{rms})_{max}$. In Figure 3 the rms value of the streamwise and spanwise vorticity are plotted across the boundary layer. Here $(\Omega x_{rms})_{max} = .77(\Omega z_{rms})_{max}$. The streamwise vorticity is very close in magnitude to the spanwise vorticity emphasizing the three dimensional nature. This also shows that streamwise vorticity can be excited by the presence of infinitesimal free-stream turbulence whereas it was previously thought that it

was due to a secondary internal instability. In Figure 4 the distribution (in the computational domain) of the Reynolds stress is plotted. The left axis is the computation domain and the right axis is the Blasius variable ($y = 5$ is the edge of the Blasius boundary layer). The maximum occurs at about 20% of the boundary layer thickness. The qualitative features shown here vary little as the Reynolds number is increased to 2000 and 3000. The function governing the Reynolds stress gets a little broader and the ratio of the streamwise to spanwise vorticity maximums drops to .58 at $R = 3000$ (compared with .77 at $R = 1000$). Otherwise the results are similar.

However these results are for zero separation in space and time. In keeping with the theme of elucidating the three-dimensional effects, results for non-zero spanwise separation will be shown. In Figure 5 the normalized $\langle u,u \rangle$ velocity auto-correlation is shown as a function of spanwise separation. This is for a fixed point in the boundary layer (a point in y which is at 26% of the boundary layer thickness). The results for $R = 1000$, 2000, and 3000 are included. ζ is normalized by the Blasius variable, therefore $\zeta = 5$ would correspond to a boundary layer thickness. The result shows a definite finite structure in the spanwise direction with increasing spanwise length scale with increasing Reynolds number. This characteristic scale is not a function of the much smaller length scale of the freestream turbulence but is governed by the unstable response of the boundary layer over a wide, but band-limited, range of three-dimensional normal modes.

In Figure 6 a similar result for the streamwise vorticity is plotted at the same location in y as the $\langle u,u \rangle$ correlation. Again there is evidence of a three-dimensional character indicative of streamwise vortices of large scale excited in the boundary layer. The typical length scale in the spanwise direction of the streamwise vorticity is also growing with increasing Reynolds number.

The main new feature of the results shown so far is the prevalent three dimensionality. Since the formulation of the problem is for free stream turbulence of

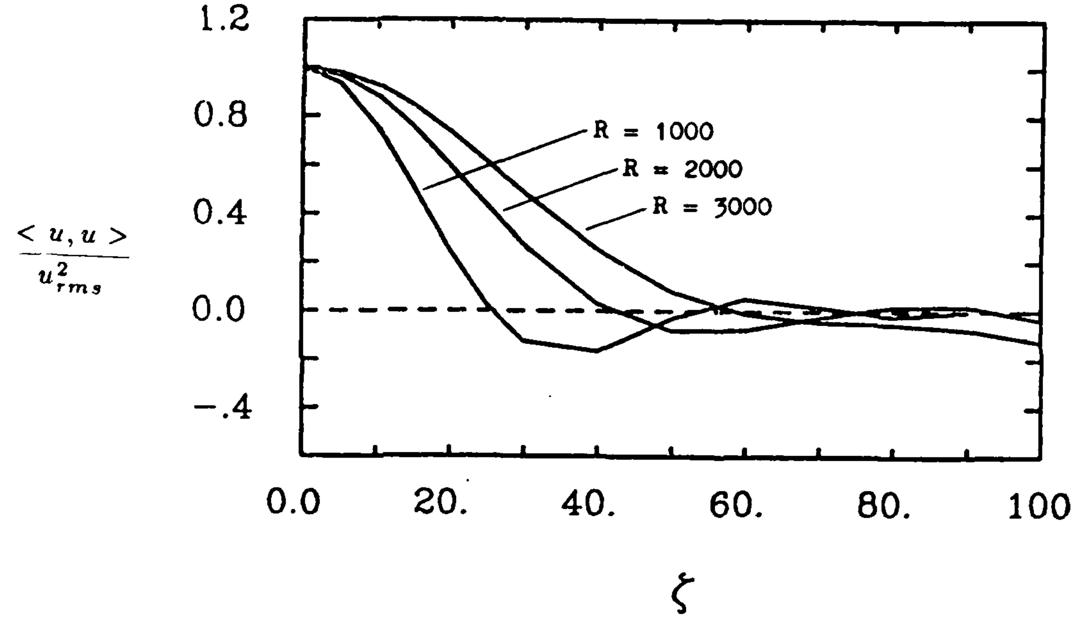

Figure 5. The autocorrelation function for the streamwise velocity perturbation as a function of the spanwise separation for $R = 1000$, 2000, 3000.

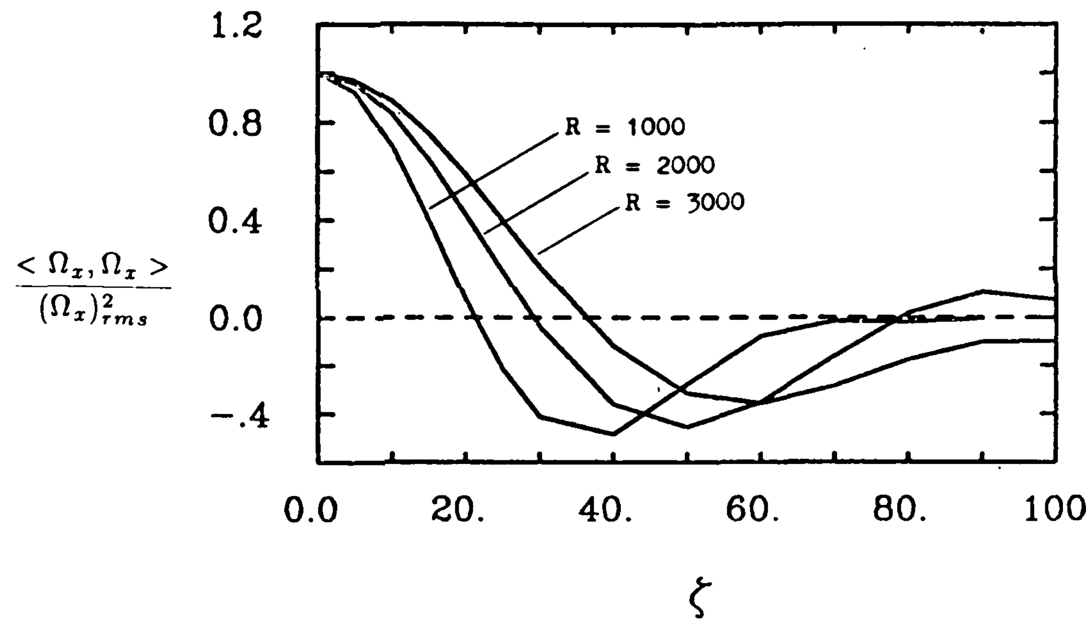

Figure 6. The autocorrelation function for the streamwise vorticity perturbation as a function of the spanwise separation for $R = 1000, 2000, 3000$.

infinitesimal amplitude, this suggests that in any real situation a three dimensional response of the nature shown will be excited. However it is instructive to ask how the solution might differ if the free stream disturbances were strictly two-dimensional. In this case the spectral integrands contain a Dirac delta function in β. Therefore only the $\beta = 0$ point of the integrand is taken resulting in a two-dimensional response in the boundary layer. The evaluation of the normalized $\langle v,v \rangle$ and $\langle u,u \rangle$ auto-correlations, for zero delay, at $R = 1000$, for the case of a two-dimensional free stream are shown in Figure 7 where the distribution of the rms u velocity is plotted. This u result is quite different from the three-dimensional result. The 2-D result is much smoother and doesn't contain the extra "hump" near the wall which is prevalent in the 3-D results. This result can be compared with the experimental measurements of Klebanoff, et al. (1962). In their experiments on natural transition they measured the rms distribution of the u velocity. Figure 8 is a reproduction from their paper. This can be compared with the three dimensional results in Figure 2 and the 2-D result in Figure 7. Although the result of Klebanoff, et al. is for a Reynolds number of 2800, Figures 2 suggests that the structure of the rms distribution varies little with Reynolds number. The presence of the extra "hump" in the result of Klebanoff, et al. suggests that there is a 3-D excitation present in their experiment. This structure is not unique to this particular figure of Klebanoff, et al. either. Most of the experimental results displayed in the paper of Klebanof, et al. have a similar structure.

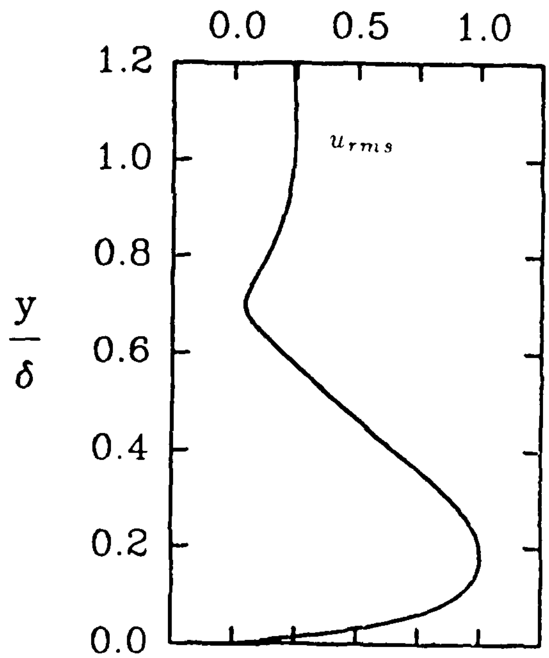

Figure 7 Effect of two-dimensional free stream disturbances on the responce. The transverse distribution of the rms streamwise velocity at R 1000.

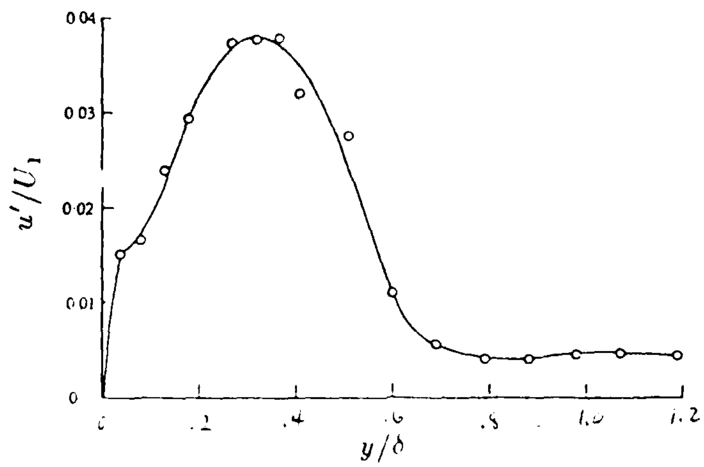

Figure 8. Distribution of intensity of u-fluctuation across the boundary layer for 'natural transition' at $R \sim 2800$. This is reproduced from Figure 34a from the experimental work of Klebanoff, et al. (1962).

Section 8. Discussion

Using the classical theory of random forcing of a linear MDOF vibrating system, the random excitation of the Blasius boundary layer has been analyzed. The Blasius boundary layer is treated as a MDOF vibrating system subjected to a random forcing function; the free stream turbulence. The main result of the present analysis is the prevalence of three dimensionality in the result. This suggests that boundary layers will have three-dimensional velocity and vorticity fields excited in them due to the presence of infinitesimal free stream turbulence (a virtual surety). Furthermore the three dimensionality is a linear phenomena, and can therefore be referred to as a primary instability rather than a secondary instability.

There does, however, remain some unclear points. The most important of these is what can be referred to as the paradox of boundary layer transition. This can be explained as follows. In Section 2 it was pointed out that for infinitesimal amplitude free stream turbulence, the terms of order ε^2 can be dropped, and then the governing equations for the perturbations arising in the boundary layer are linear. Taking the limit of these equations as $y \to \infty$ should result in the governing equations for the free stream disturbances. Unfortunately the eigenvalues of this set of equations form a continuous spectrum and the eigenfunctions are harmonic. Since any set of disturbances will essentially be expanded in terms of these eigenfunctions, this set of equations will not support disturbances with discrete eigenvalues. Therefore this free stream turbulence will not be capable of exciting Tollmien-Schlichting waves in the boundary layer. However, it is generally accepted that Tollmien-Schlichting waves do actually exist in experiments, and that linear theories are in agreement with this. This is the paradox of boundary layer transition.

This was overcome in the present study by assuming that the flat plate was situated in a large wind tunnel and therefore the set of equations in the free stream is approximately correct. However, in a more complete analysis, it may be necessary to

consider the free stream disturbances to be governed by a non-linear equation, say the Karman-Howarth equation for isotropic turbulence, and then treat this as an outer solution and the linear boundary layer perturbations as an inner solution.

Another interesting point which is not obvious but does arise in the analysis is the effect of homogeneity, in the streamwise direction, of the free stream turbulence. Two factors were considered in developing an expression for the autocorrelation function governing the free stream turbulence; (a) the expression should be general and capable of approximating properties known to occur in experiments, and (b) since the mean flow in the free stream is constant, turbulence production will not occur, therefore the free stream turbulence will decay in x. This resulted in the expression (3.8). It is worthwhile to consider the limiting case of homogeneous, in x, free stream turbulence. Surprisingly, this problem is insoluble using the present analysis. The integral in (5.7), for example, will be over α_1 instead of α_1 and α_2. When the neutral curve is encountered in the integration there will be a double pole on the real α_1 axis, which is non-integrable. The analogy of this is the SDOF equation

$$\frac{d^2\varphi}{dt^2} + \omega_0^2\varphi = f(t) \tag{8.1}$$

If $f(t) = \sin(\omega t)$ there is a resonance at $\omega = \omega_0$ where the solution is unbounded. However if $f(t) = e^{-\mu t}\sin(\omega t)$ a bounded solution can be obtained. For example the particular solution would be

$$\varphi_1(t) = e^{-\mu t}(A\sin\omega t + B\cos\omega t) \tag{8.2a}$$

$$A = \frac{\mu^2 - \omega^2 + \omega_0^2}{(\mu^2 - \omega^2 + \omega_0^2)^2 + (2\mu\omega)^2} \tag{8.2b}$$

$$B = \frac{2\mu\omega}{(\mu^2 - \omega^2 + \omega_0^2)^2 + (2\mu\omega)^2} \tag{8.2c}$$

A similar result occurs when $f(t)$ is a random function. If $f(t)$ is random and stationary in time, a solution cannot be found because the transfer function has a double pole on the real ω axis. However, if $f(t)$ is random and non-stationary (decaying in t, for example) then the integrals exist and a statistical solution may be found.

This is essentially what happens in the boundary layer analysis. The neutral curve is a singularity of the linear theory if the free stream disturbances are homogeneous in the streamwise direction. However the governing equations for the free stream turbulence have a parabolic term, proportional to $1/R$, which suggests a variation in amplitude of the solution, and additionally, on physical grounds it is expected that free stream turbulence, in the absence of a mean shear, will decay. Therefore the use of a set of disturbances with decaying amplitude is the proper choice. But this nevertheless leads to speculation on the effect of non-linearity on the singularity.

The SDOF equation can again be used as an analogy,

$$\varphi'' + \omega_0^2\varphi + \varphi^3 = \varepsilon f(t) \tag{8.3}$$

This is the well known undamped Duffing equation (Nayfeh, Chapter 9). To find solutions for ω near ω_0 one defines a detuning parameter $\omega = \omega_0 + \varepsilon^{2/3}\sigma$. Then it can be formally shown that φ must be expanded in a series

$$\varphi(t) = \sum_{n=1}^{\infty} \varepsilon^{n/3}\varphi_n(t) \tag{8.4}$$

This shows that a forcing function of order ε excites a response of order $\varepsilon^{1/3}$ (as $\varepsilon \to 0$) near resonance. A similar argument can be used for the boundary layer. A standard Stuart-Watson expansion for the non-linear modes near the neutral curve suggests that the amplitude correction to the linear eigenvalues is of the order of the amplitude squared. Therefore in order to bring in the non-linear terms correctly to overcome the singularity, the solution in the boundary layer should be expanded in powers of $\varepsilon^{1/3}$ where ε is the amplitude of the free stream disturbances. The main

implication is that disturbances in the free stream of order ϵ, which are homogeneous in x, with dispersion properties near those of the neutral curve will excite a response in the boundary layer of order $\epsilon^{1/3}$ (as $\epsilon \to 0$). The details of such a non-linear analysis are not given here as the complete analysis will be the subject of a future paper.

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Appendix A

The matrix $[G_1]$, first used in equation (4.14) can be derived in the following manner. It is a result of the Chebyshev discretization of the boundary condition function $g_1(x,y,z,t)$.

If

$$y = 2(1 + Y)/(1 - Y) \tag{A.1}$$

and

$$\exp(-y) = \sum_{n=0}^{\infty}{}' d_n T_n(Y) \tag{A.2}$$

and

$$y\exp(-y) = \sum_{n=0}^{\infty}{}' e_n T_n(Y) \tag{A.3}$$

then

$$g_{1n}(x,z,t) = \sum_{j=0}^{N}{}' (-1)^j q^j (x,z,t) (d_n + e_n(1 - j^2)) \tag{A.4}$$

therefore the matrix $[G_1]$ has entries

$$[G_1]_{i,j} = (-1)^j (d_i + e_i(1 - j^2)) \quad \text{for} \quad i = 0, 1, \ldots, N; \quad j = 0, 1, \ldots, N \tag{A.5}$$

The matrix $[G_2]$ arises from the Chebyshev discretization of the boundary condition function $g_2(x,y,z,t)$ and it has entries

$$[G_2]_{i,j} = (-1)^j d_i \quad \text{for} \quad i = 0, 1, \ldots, N \quad \text{and} \quad j = 0, 1, \ldots, N \tag{A.6}$$

where d_i is given in (A.2).

The matrix $[D_1]$ is a discretization of the operation md/dY. The metric $m(Y)$ has the Chebyshev expansion,

$$m(Y) = \frac{1}{2} m_0 + m_1 T_1(Y) + m_2 T_2(Y) \tag{A.7}$$

where $m_0 = 3/4$, $m_1 = -1/2$, and $m_2 = 1/8$. If we define a pentadiagonal matrix $[M]$ such that

$$[M] = \begin{bmatrix} M_0 & 2M_1 & 2M_2 & 0 & --- & & \\ M_1 & M_0 + M_2 & M_1 & M_2 & 0 & --- & \\ M_2 & M_1 & M_0 & M_1 & M_2 & 0 & --- \\ 0 & M_2 & M_1 & M_0 & M_1 & M_2 & --- \\ & - & - & - & - & - & \\ & - & - & - & - & - & \\ & - & - & - & - & - & \end{bmatrix} \tag{A.8}$$

then the matrix $[D_1]$ has entries

$$[D_1]_{i,j} = \sum_{K=0}^{j} M_{i,k}(1 - (-1)^{j+k})j \quad \text{for } i = 0,1, \ldots, N \text{ and } j = 0,1, \ldots, N \tag{A.9}$$

Appendix B

Consider the matrix $[D_p(\alpha)]$, which depends on the scalar α in the following way,

$$D_p(\alpha) = \sum_{j=0}^{p} C_j \alpha^{p-j} \tag{B.1}$$

where the C_j are square complex matrices. The determinant of $D_p(\alpha)$ is defined as $\Delta(\alpha)$, and its derivative is

$$\Delta^{(1)}(\alpha) = \frac{d}{d\alpha} \Delta(\alpha)$$

It is necessary in the present work to evaluate the derivative at (simple) latent roots, where $\Delta(\alpha_k) = 0$.

The classical expression for the derivative of a determinant can be constructed in the following way. Define $D_{*j}(\alpha)$ as the jth column of $D_p(\alpha)$, and $D_p^{(1)}(\alpha)$, as the derivative of $D_p(\alpha)$ with respect to α. Then if N is the order of the matrix $D_p(\alpha)$, the derivative of the determinant of $D_p(\alpha)$ can be expressed as

$$\Delta^{(1)}(\alpha) = \sum_{j=1}^{N} \Delta_j(\alpha)$$

where $\Delta_j(\alpha)$ is the determinant of a matrix whose jth column is $D_{*j}^{(1)}(\alpha)$ and the remaining columns are those of $D_p(\alpha)$. Unfortunately this expression requires an operation count of $0(N^4)$ for each point α_k, which is clearly prohibitive.

With a slight rearrangement of the matrices, the local derivative of a determinant can be evaluated in a straightforward and efficient manner. The idea is to arrange $D_p(\alpha)$ in the following Taylor form

$$D_p(\alpha) = \sum_{j=0}^{N} F_j(\alpha - \alpha_k)^j \tag{B.4}$$

where

$$F_j = \frac{1}{j^i} \frac{d^j}{d\alpha^j} D_p(\alpha)\bigg|_{\alpha=\alpha_k} \tag{B.5}$$

However, only the first two matrices need to be retained to determine $\Delta^{(1)}(\alpha_k)$ exactly.

$$D_p(\alpha) = F_0 + F_1(\alpha - \alpha_k) + \ldots \tag{B.6}$$

Obviously, F_0 is singular since $\Delta(\alpha_k) = 0$, therefore F_0 can be brought to the form

$$A = \tilde{F}_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & - & - & - & 0 \\ 0 & x & & x & & x & & x \\ 0 & & x & & x & & & x \\ - & & & & - & & & x \\ - & & & & & - & & x \\ 0 & & & & & & - & x \end{bmatrix}$$

The first row and first column of F_0 are brought to zero and the remaining $N - 1$ by $N - 1$ minor is brought to upper triangular form, with the same operations performed on F_1. If $B = \tilde{F}_1$ (F_1 after operations), then

$$\Delta^{(1)}(\alpha_k) = (b_{11}) \cdot (a_{22})(a_{33}) \ldots (a_{NN}) \tag{B.8}$$

The derivative of the determinant is the product of the determinant of the principal minor of A times the upper left entry of B.

It should also be apparent that in the limit as $\alpha \to \alpha_k$, there is only one non-zero entry in the cofactor matrix,

$$\lim_{\alpha \to \alpha_k} C(\alpha) = \begin{bmatrix} x & 0 & 0 & 0 \\ 0 & & 0 & \\ 0 & & & \\ 0 & & & \end{bmatrix} \tag{B.9}$$

where $c_{11} = (a_{22})(a_{33}) \ldots (a_{NN})$ is the determinant of the principal minor of A. In the computation, however, (B.8) appears as a divisor to (B.9), therefore the determinant of the principal minor of A need not be calculated. The above approach results in an efficient method for evaluating the necessary inverse Fourier transforms.

A generalization of this result which includes singular and non-singular points and higher derivatives is contained in Bridges and Vaserstein (1985).

Appendix C

Theorem: The secondary operator $\mathbf{L_2}$ governs the streamwise and spanwise velocity perturbations in the Orr-Sommerfeld problem,

$$\mathbf{L_2}\Upsilon \stackrel{\text{def}}{=} \frac{d^2\Upsilon}{dy^2} + \left[iR(\omega - \alpha U(y)) - (\alpha^2 + \beta^2)\right]\Upsilon \tag{C.1}$$

where $\omega, \beta, R \in \mathbf{R}$ with $R > 0$, $\alpha \in \mathbf{C}$ is the "spatial" eigenvalue.

If Υ is square integrable in $y \in [0,\infty)$, $\Upsilon(0) = 0$, $\Upsilon(y) \to 0$ as $y \to \infty$, and $U(y)$ is non-negative and bounded for all y, then the spatial eigenvalues of $\mathbf{L_2}$ are stable, i.e. $Im(\alpha) > 0$, for any $R > 0$.

Proof: Set $\mathbf{L_2}\Upsilon = 0$ and multiply thru by Υ^*, the complex conjugate of Υ and integrate over y.

$$-\int_0^\infty \left|\frac{d\Upsilon}{dy}\right|^2 dy + (i\omega R - \alpha^2 - \beta^2)\int_0^\infty |\Upsilon|^2 dy - i\alpha R\int_0^\infty U(y)|\Upsilon|^2 dy = 0 \tag{C.2}$$

With $U(y)$ non-negative and bounded for all y the three integrals may be represented by positive definite constants

$$-a_3 + (i\omega R - \alpha^2 - \beta^2)a_1 - i\alpha R a_2 = 0 \tag{C.3}$$

which may be solved for α,

$$\alpha = -\frac{iRa_2}{2a_1}\left\{1 \pm \sqrt{1 + (\frac{2a_1\beta}{a_2R})^2 + \frac{4a_1a_3}{a_2^2R^2} - \frac{i4\omega a_1^2}{a_2^2R}}\right\} \tag{C.4}$$

But the square root of a complex number is

$$\sqrt{z} = \sqrt{x + iy} = \pm\left\{\sqrt{\frac{1}{2}(|z| + x)} + i[sign(y)]\sqrt{\frac{1}{2}(|z| - x)}\right\} \tag{C.5}$$

so that

$$\sqrt{1 + (\frac{2a_1\beta}{a_2R})^2 + \frac{4a_1a_3}{a_2^2R^2} - \frac{i4\omega a_1^2}{a_2^2R}} = a_4 - ia_5 \tag{C.6}$$

where a_4 and a_5 are positive definite constants and the sign of ω is ignored as it has no effect on the outcome. Then from (C.4)

$$\alpha = -\frac{iRa_2}{2a_1}(1 \pm (a_4 - ia_5)) \tag{C.7}$$

Choosing only the downstream directed wave ($Re(\alpha) > 0$),

$$\alpha_+ = \frac{a_2a_5}{2a_1}R + i(a_4 - 1)\frac{a_2R}{2a_1} \tag{C.8}$$

Thus the eigenvalue is stable is $a_4 \geq 1$ and unstable is $a_4 < 1$. Now

$$a_4 = \sqrt{\frac{1}{2}|z| + \frac{1}{2}\left\{1 + (\frac{2a_1\beta}{a_2 R})^2 + \frac{4a_1a_3}{a_2^2R^2}\right\}} \qquad (C.9)$$

where

$$|z| = \left\{\left\{1 + (\frac{2a_1\beta}{a_2R})^2 + \frac{4a_1a_3}{a_2^2R^2}\right\}^2 + \frac{16\omega^2a_1^4}{a_2^4R^2}\right\}^{\frac{1}{2}} \qquad (C.10)$$

Clearly in the limit as $R \to \infty$, $a_4 \to 1$, and for all finite values of R, $a_4 > 1$. This proves the theorem.

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER
2882

2. GOVT ACCESSION NO.
AD-A163606

3. RECIPIENT'S CATALOG NUMBER

4. TITLE (and Subtitle)
A STATISTICAL ANALYSIS OF THE EFFECT OF FREESTREAM TURBULENCE OF THE BLASIUS BOUNDARY LAYER

5. TYPE OF REPORT & PERIOD COVERED
Summary Report - no specific reporting period

6. PERFORMING ORG. REPORT NUMBER

7. AUTHOR(s)
Thomas J. Bridges and Philip J. Morris

8. CONTRACT OR GRANT NUMBER(s)
DAAG29-80-C-0041
DMS-8210950, Mod. 1

9. PERFORMING ORGANIZATION NAME AND ADDRESS
Mathematics Research Center, University of Wisconsin
610 Walnut Street
Madison, Wisconsin 53706

10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
Work Unit Number 2 - Physical Mathematics

11. CONTROLLING OFFICE NAME AND ADDRESS
(See Item 18 below)

12. REPORT DATE
October 1985

13. NUMBER OF PAGES
52

14. MONITORING AGENCY NAME & ADDRESS(If different from Controlling Office)

15. SECURITY CLASS. (of this report)
UNCLASSIFIED

15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)
Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES
U. S. Army Research Office
P. O. Box 12211
Research Triangle Park
North Carolina 27709

National Science Foundation
Washington, DC 20550

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)
boundary layers, turbulence, statistics, Chebyshev polynomials

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)
This paper considers the flow past a flat plate. When the flow is uniform the Blasius boundary layer forms on the plate. Herein the usual uniform free stream is perturbed by a small amplitude field of random three-dimensional turbulence. Using a numerical technique the statistical response in the boundary layer due to the turbulence in the free stream is computed in terms of auto- and cross-correlation functions. The field variables are expanded in finite Chebyshev series and the space-time correlations of these

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UNCLASSIFIED
SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

20. ABSTRACT (Continued)

functions are found by forming correlation matrices with the vectors of random coefficients of the Chebyshev expansions.

The main result of the evaluations is the fact that three dimensional random infinitesimal free stream turbulence produces a full three dimensional response in the boundary layer. The excited field includes the presence of streamwise vorticity as well which was previously thought to be due to secondary instability.

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